Price fluctuations

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Return levels in the POT models

- The $1-p$-quantile, based on the generalized Pareto model:

$$z_p = \begin{cases} u + \frac{\tilde{\sigma}}{\xi}\left[\left(\frac{p}{\eta}\right)^{-\xi} - 1\right], & \xi \neq 0 \\ u - \tilde{\sigma}\log\left(\frac{p}{\eta}\right), & \xi = 0 \end{cases}$$

where $\eta = P(X > u)$, $\hat{\eta} = \frac{n_u}{n}$. n_u is the number of such observations in the sample that exceed u

- This is the value that is exceeded on average once in every $1/p$ observations
- If we observe annually on average n_y maxima over the threshold, then the $\frac{1}{T*n_y}$ quantile is returning once in every T years on average.
- If $\xi < 0$, the estimated upper endpoint of the distribution is $u - \tilde{\sigma}/\xi$.

Threshold selection

- The mean excess plot: We plot the average of $X - u$ for different thresholds u (for those observations, which fulfill $X > u$) as a function of u.
- If the Pareto model is true, this curve is near to linear. The explanation of its properties may be difficult due to its wild fluctuations near the maximum of the observations.
- An alternative: let us consider the estimated values of the parameters for different thresholds.

Example: S&P 500, daily losses

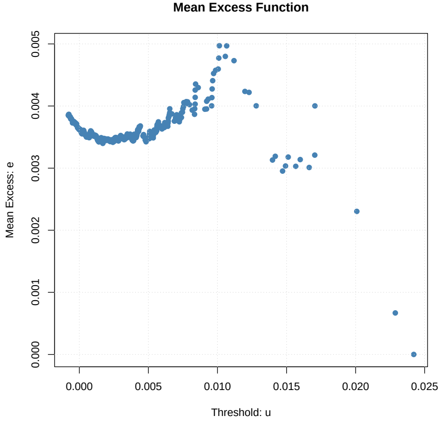


Figure 1: Mean excesses as the function of the threshold

- Daily losses in %
- 1960–1987
- Threshold: 1.5%
- MLE: $\xi = 0.177, \sigma = 0.415$
- 40 years return level: 6.57, conf. int: (4.9;10)
- Data for the period 2002-2005
- Threshold: 1%
- MLE: $\xi = -0.31, \sigma = 0.62$
- 10 years return level: 2.8
- The largest observed loss was on 13.10.2008: 9.05%

Multivariate POT models

- It would be good to use more data and real observations
- Classic model: the observations should exceed the threshold in all coordinates
  - Margins: GPD
  - Certain parametric MGEV models may be transferred
  - R package EVD can be used
- But: the used number of observations is low

Alternative definition (BGPD II)

- It considers every observation that exceeds the threshold in at least one coordinate
- Formally: $\underline{Y} = (Y_1, ..., Y_d)$ is a random vector, $\underline{u} = (u_1, ..., u_d)$ a suitably high threshold $\underline{X} = \underline{Y} - \underline{u} = (Y_1 - u_1, ..., Y_d - u_d)$ the exceedances
- H is a *d*-dimensional GPD (MGPD) if:

$$H(x_1, \ldots, x_d) = \frac{-1}{\log G(0, \ldots, 0)} \log \frac{G(x_1, \ldots, x_d)}{G(x_1 \wedge 0, \ldots, x_d \wedge 0)},$$

  where G has MGEV distribution
- We get standard Fréchet marginals by the transform

$$t_i = t_i(x_i) = \frac{-1}{\log G_{\xi_i, \mu_i, \sigma_i}(x_i)} = (1 + \xi_i(x_i - \mu_i)/\sigma_i)^{1/\xi_i},$$

  where $1 + \xi_i(x_i - \mu_i)/\sigma_i > 0$ and $\sigma_i > 0$, $i = 1, ..., d$.

Alternative definition (BGPD II)

- The MGPD density (if exists)

$$h(\underline{x}) = \frac{\partial H}{\partial x_1 \dots \partial x_d}(\mathbf{x}) = \frac{\partial}{\partial x_1 \cdots \partial x_d}\left(1 - \frac{\log G(\underline{x})}{\log G(\underline{0})}\right)$$
$$= \frac{\prod_{i=1}^{d} t_i'(x_i)}{V\Big(t_1(0), \dots, t_d(0)\Big)} \times \frac{\partial V}{\partial t_1 \cdots \partial t_d}\Big(t_1(x_1), \dots, t_d(x_d)\Big).$$

- The margins will not be GPDs, but $Z_i | Z_i > 0$ is GPD for the 1-dimensional margins
- There are just a few models which can be identified
- Estimation: with the R package `mgpd`, but it is not easy in more than two dimensions
- Comparison: bootstrap simulations show that those estimators, which are based on more data are indeed more reliable

Comparison of the models

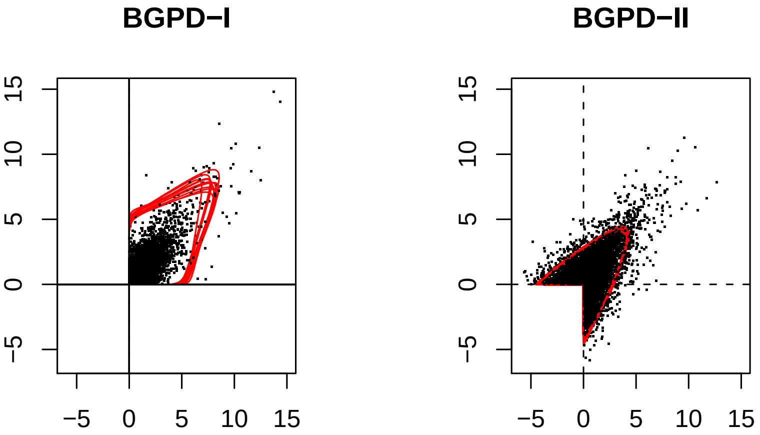


Figure 2: Coverage sets for simulated data

Comparing two 2D models

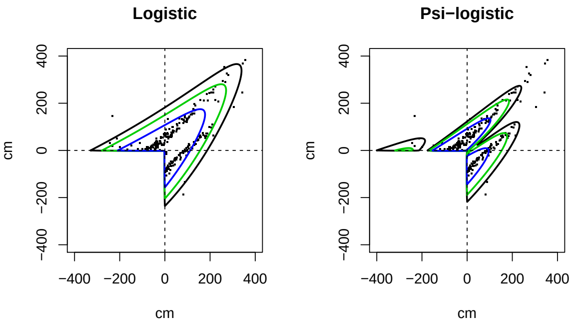


Figure 3: Comparing symmetric and asymmetric models

Psi-logistic model: $A(\Psi(t)) = A(t + f(t))$, where $f_{\psi_1,\psi_2}(t) = \psi_1(t(1-t))^{\psi_2}$, if $t \in [0, 1]$, $\psi_1 \in \mathbb{R}$ and $\psi_2 \geq 1$ are asymmetry parameters

A new representation

- Here we assume exponential margins
- Let $Y_i = Z - \max(X_1, ..., X_d) + X_i$, where Z has unit exponential distribution, and $\underline{X}$ is independent from Z.Then Y has an MGPD II distribution, with unit exponential margins conditionally for $Y_i > 0$.
- Moreover, every such MGPD vector can be expressed in this way
- For certain generators with independent components,the density can be calculated
- ML method can be used for estimation

Estimation methods, based on upper order statistics

Assumption: the observations come from F, which is in the MDA of a GEV distribution G.

- Hill estimator
- Pickands estimator
- Method of moments
- ML estimator for the exponential regression

Hill estimator

- For a heavy tailed distribution: $P(X > z) = z^{-\alpha} L(z)$.
- The distribution of $\log(X)$: $P(\log X > u) = e^{-\alpha u} L(e^u)$.
- Quantile function: $Q(1 - p) = p^{-\gamma} L^*(1/p)$, thus

$$\log Q(1 - p) = -\gamma \log p + \log L^*(1/p).$$

- The ordered sample of $X_1, X_2, \ldots, X_n$ is denoted by $X_1^* \le X_2^* \le \cdots \le X_n^*$.
- The elements of the ordered sample are consistent estimators for the respective quantiles.
- Plot: $\log X^*_{n-j+1}$ vs $\log \frac{j}{n+1}$.
- It is asymptotically linear with steepness γ.

Hill estimator/2

- Estimation of the steepnes:

$$\frac{\frac{1}{k}\sum_{j=1}^{k}\left(\log(X^*_{n-j+1}) - \log(X^*_{n-k+1})\right)}{-\frac{1}{k}\sum_{j=1}^{k}\left(\log(\frac{j}{n+1}) - \log(\frac{k}{n+1})\right)}$$

- The denominator is near to 1, if k is large
- Thus the Hill estimator:

$$H_{k,n} = \frac{1}{k}\sum_{j=1}^{k}\left(\log(X^*_{n-j+1}) - \log(X^*_{n-k+1})\right).$$

- See Embrechts et al., p. 331 for another motivation
- Properties:
  - depends on k
  - k small: large variance
  - k large: substantial bias
  - A compromise must be found. Not too robust!

Methods for choosing k

- $\hat{\xi}$ is a consistent estimator for the tail index only if $k \to \infty$ and $k/n \to 0$ as $n \to \infty$.
- The double bootstrap method proposes k by minimizing the asymptotic mean squared error, based on bootstrap resampling.
- Another method minimizes the distance between the tail of the empirical distribution function and the fitted Pareto distribution with the estimated tail index parameter. For the minimization the Kolmogorov-Smirnov distance of the quantiles can be used.

Example: GBP vs DM daily return

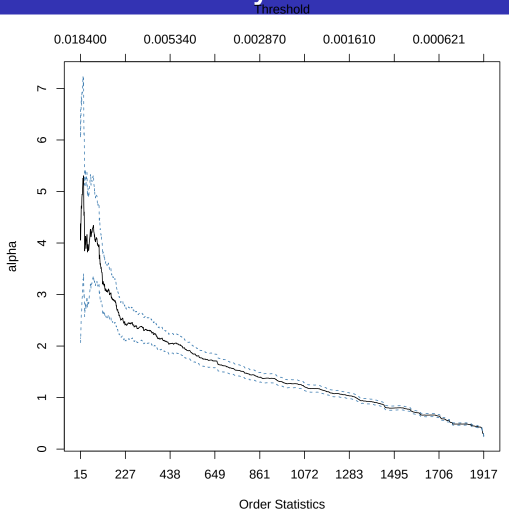


Figure 4: The Hill estimator and its confidence bounds as a function of the threshold

Point processes

- A useful tool, it is easy to generalise the models with its help
- The sample is a sequence of random points, their number is n
- $N(B)$: the number of observations falling into the Borel set B.
- The extremes are interesting for us: we usually deal with points falling into (u, ∞)
- Poisson point process: generalisation of the one dimensional Poisson process. Its properties:
  - $N(A)$ and $N(B)$ are independent for any disjoint sets A and B
  - If $\Lambda \geq 0$ is a given measure over the Borel sets (intensity), $N(B)$ has Poisson distribution with parameter $\Lambda(B)$
  - The process is called homogeneous if Λ is constant times the Lebesgue measure (taken over the bounded Borel sets B)

Statistical inference for point processes

- The intensity λ is supposed to come from a parametric family

$$L(\vartheta; x_1, \ldots, x_n) = \exp\{-\Lambda(A; \vartheta)\} \prod_{i=1}^{n} \lambda(x_i; \vartheta)$$

  gives the likelihood if we have observations from a region A.
- Relation to the GPD model: the intensity measure for exceedances beyond u_n is

$$\Lambda(A) = (t_2 - t_1) \left[1 + \xi \frac{z}{\sigma}\right]^{-1/\xi}$$

  for a region $(t_1; t_2) \times (z, \infty)$ where $z > u_n$
- This approach allows for more complex models, e.g. time dependence of the parameters.

Max-stable processes

- Let $T \subset \mathbb{R}^d$ be an index set (in case of the basic definition these were precipitation measuring stations), then $\{Y_t : t \in T\}$ is a max-stable process if and only if its coordinates are max-stable.
- Example: let (r_i, s_i) be a Poisson point process over the set $(0, \infty) \times S$ with intensity measure

$$\frac{dr}{r^2} dH(\omega)$$

where S is an arbitrary measurable set, and H a measure over S.

The construction of Smith (1990)

- Let (r_i, s_i) be a Poisson point process over the set $(0, \infty) \times S$ with intensity measure

$$\frac{dr}{r^2} dH(\omega)$$

- Let f be such that $\int_S f(s,t)dH(s) = 1$ for all t and

$$Y_t := \max_i \{r_i f(s_i, t)\}, t \in T$$

- r_i is the strength of the ith storm and s_i is its location.

$$P(Y_t < y_t, \ \forall t \in T) = \exp\left\{ - \int_S \max_t \left\{ \frac{f(s,t)}{y_t} \right\} dH(s) \right\}.$$

- This implies:
  - the margin of Y is unit Frechet
  - Y is max-stable

Examples

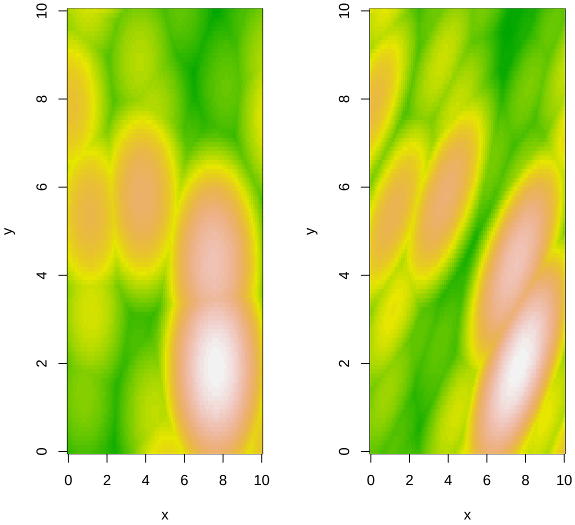
- $|T| = 1$: one dimensional max-stable distribution
- $T = \{1, 2\}$, $S = [0, 1]$, H is the Lebesgue measure, for $0 < \alpha < 1$

$$f(s,t) = \begin{cases} (1-\alpha)s^{-\alpha}, & \text{if } t = 1 \\ (1-\alpha)(1-s)^{-\alpha}, & \text{if } t = 2 \end{cases}$$

is just the 2 dimensional logistic model

- Gaussian extreme-value process: $f(s,t)$ is the density function of the normal distribution with mean s and covariance-matrix Σ as a function of $s - t$

Example: simulated Smith-type extremal processes



Model-fitting

- The estimation of the 1 dimensional margins
- Estimation of the 2 dimensional dependence (extremal index) $\vartheta(z_1 - z_2)$, where

$$P(Y(z_1) < y, Y(z_2) < y) = P(Y(z_1) < y)^{\vartheta(z_1 - z_2)}.$$

- For parametric – e.g. Gauss - models approximate (pairwise) maximum likelihood estimation can be calculated; we come back to this question later

References

- Danielsson, J., de Haan, L., Peng, L. and de Vries, C. G. (2001). Using a bootstrap method to choose the sample fraction in tail index estimation
- Danielsson, J., Ergun, L. M., De Haan, L. and de Vries, C. G. (2016). Tail Index Estimation: Quantile Driven Threshold Selection
- R.L. Smith (1990) Max-Stable Processes and Spatial Extremes. `www.stat.unc.edu/postscript/rs/spatex.pdf`
- Schlather, M. and Tawn, J. (2003) A dependence measure for multivariate and spatial extreme values: Properties and inference.
- Rootzén, H. and Tajvidi, N. (2006) The multivariate generalized Pareto distribution. Bernoulli 12, p.917-930.
- A. Kirilouk et al. (2018) Peaks Over Thresholds Modeling With Multivariate Generalized Pareto Distributions.
- Rakonczai, P.: Multivariate Threshold Models with Applications to Wind Speed Data (Ph.D. thesis, 2012)